PEJABAT PENGURUSAN DAN KREATIVITI PENYELIDIKAN
RESEARCH CREATIVITY AND MANAGEMENT OFFICE
UNIVERSITI SAINS MALAYSIA

Ruj. Kami : FPP 2006/203 (P2642)
Tarikh : 20 April 2009

Dr. Roslan Hasni @ Abdullah
Pusat Pengajian Sains Matematik
Universiti Sains Malaysia

Tuan,

Laporan Akhir Projek Penyelidikan USM Jangka Pendek
Tajuk Projek : "An Attempt To Classify Bipartite Graphs by Their Chromatic Polynomial"
No. Akaun : 304/PMATHS/637053

Dengan hormatnya perkara di atas dirujuk.

2. Terlebih dahulu saya ucapkan terima kasih di atas satu salinan laporan akhir untuk projek penyelidikan USM jangka pendek seperti tajuk di atas. Bersama ini dilampirkan komen penilaian daripada Pemangku Dekan Penyelidikan Pelantar Teknologi Maklumat & Komunikasi untuk perhatian tuan.

3. Seterusnya walaupun projek ini telah selesai, Jabatan Bendahari telah dinasihatkan untuk menangguhkan penutupan akaun projek kepada **30 April 2009.** Tempoh ini diberi untuk membolehkan penjelasan semua urusan tuntutan dan bayaran yang telah dikomitkan di dalam tempoh projek. Walau bagaimanapun, tuan dinasihatkan supaya tidak mengeluarkan borang-borang pesanan baru di dalam tempoh ini.

4. Selanjutnya sila ambil perhatian terhadap perkara-perkara berikut sekiranya berkaitan :

(i) semua penerbitan harus merakamkam penghargaan kepada geran penyelidikan jangka pendek dan tuan dipohon mengemukakan satu salinan ke pejabat ini; dan

(ii) pihak kami akan mengagihkan semula peralatan yang telah dibeli menggunakan peruntukan geran ini seandainya terdapat penyelidik lain yang memerlukan peralatan tersebut.

5. Harap maklum, projek ini dianggap telah selesai dengan jayanya.

Sekian, terima kasih.

"BERKHIDMAT UNTUK NEGARA"
'Memastikan Kelestarian Hari Esok'

Yang menjalankan tugas,

(AMRAN OTHMAN)
Penolong Pendaftar

HAR/SFM/SRMS

s.k. Prof. Madya Bahari Belaton
Pemangku Dekan Penyelidikan
Pelantar Teknologi Maklumat & Komunikasi
Pejabat Pelantar Penyelidikan

Prof. Madya Ahmad Izani Md. Ismail
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Prof. Madya Norhashidah Hj. Mohd. Ali
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Pusat Pengajian Sains Matematik

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Pegawai Sains
Pelantar Teknologi Maklumat & Komunikasi
Pejabat Pelantar Penyelidikan

Tuan Haji Mohd Pisol Ghadzali Ketua Pustakawan Perpustakaan Hamzah Sendut 1	Disampaikan satu salinan laporan akhir projek untuk simpanan Perpustakaan
Puan Ansuya a/p Narhari Penolong Bendahari Unit Kumpulan Wang Penyelidikan Jabatan Bendahari	Sila ambil tindakan menutup akaun projek pada **30 April 2009** dan sila kemukakan satu salinanpenyata kewangan terakhir ke pejabat ini

AO/yu...laporanakhirinduk (20 April 2009)

UNIVERSITI SAINS MALAYSIA

LAPORAN AKHIR PROJEK PENYELIDIKAN JANGKA *PENDEK*

FINAL REPORT OF SHORT TERM RESEARCH PROJECT

Sila kemukakan laporan akhir ini melalui Jawatankuasa Penyelidikan di Pusat Pengajian dan Dekan/Pengarah/Ketua Jabatan kepada Pejabat Pelantar Penyelidikan

1. Nama Ketua Penyelidik:
Name of Research Leader

☐ Profesor Madya/ *Assoc. Prof.* ☑ Dr./ *Dr.* ☐ Encik/Puan/Cik *Mr/Mrs/Ms*

2. Pusat Tanggungjawab (PTJ):
School/Department

Pusat Pengajian Sains Matematik

3. Nama Penyelidik Bersama:
Name of Co-Researcher

4. Tajuk Projek:
Title of Project

An Attempt to Classify Bipartite Graphs by Their Chromatic Polynomial

5. Ringkasan Penilaian/*Summary of Assessment*:

	Tidak Mencukupi *Inadequate* 1	2	Boleh Diterima *Acceptable* 3	Sangat Baik *Very Good* 4	5
i) **Pencapaian objektif projek:** *Achievement of project objectives*	☐	☐	☑	☐	☐
ii) **Kualiti output:** *Quality of outputs*	☐	☐	☑	☐	☐
iii) **Kualiti impak:** *Quality of impacts*	☐	☐	☑	☐	☐
iv) **Pemindahan teknologi/potensi pengkomersialan:** *Technology transfer/commercialization potential*	☑	☐	☐	☐	☐
v) **Kualiti dan usahasama :** *Quality and intensity of collaboration*	☑	☐	☐	☐	☐
vi) **Penilaian kepentingan secara keseluruhan:** *Overall assessment of benefits*	☐	☐	☑	☐	☐

6. Abstrak Penyelidikan

(Perlu disediakan di antara 100 - 200 perkataan di dalam **Bahasa Malaysia dan juga Bahasa Inggeris**. Abstrak ini akan dimuatkan dalam Laporan Tahunan Bahagian Penyelidikan & Inovasi sebagai satu cara untuk menyampaikan dapatan projek tuan/puan kepada pihak Universiti & masyarakat luar).

Abstract of Research

(An abstract of between 100 and 200 words must be prepared in Bahasa Malaysia and in English). This abstract will be included in the Annual Report of the Research and Innovation Section at a later date as a means of presenting the project findings of the researcher/s to the University and the community at large)

Lihat Lampiran.

7. Sila sediakan laporan teknikal lengkap yang menerangkan keseluruhan projek ini. [Sila gunakan kertas berasingan]

Applicant are required to prepare a Comprehensive Technical Report explaning the project. (This report must be appended separately)

Sila rujuk lampiran

Senaraikan kata kunci yang mencerminkan penyelidikan anda:

List the key words that reflects your research:

Bahasa Malaysia	*Bahasa Inggeris*
Polinomial Kromatik	Chromatic Polynomial
Keunikan Kromatik	Chromatic Uniqueness
Graf Dwi-Partisi	Bipartite Graph

8. Output dan Faedah Projek

Output and Benefits of Project

(a) * **Penerbitan Jurnal**

Publication of Journals

(Sila nyatakan jenis, tajuk, pengarang/editor, tahun terbitan dan di mana telah diterbit/diserahkan)

(State type, title, author/editor, publication year and where it has been published/submitted)

(1) Jurnal, A Family of Chromatically Unique Bipartite Graph, H. Roslan and YH Peng, Far East Journal of mathematics (accepted for publication 13/8/2007)

(2) Jurnal, Chromatic Uniqueness of Certain Bipartite Graphs With Six Edges Deleted, H. Roslan and YH Peng, Thai Mathematics Journal (under review, submitted 7/11/2008)

(b) **Faedah-faedah lain seperti perkembangan produk, pengkomersialan produk/pendaftaran paten atau impak kepada dasar dan masyarakat.**
State other benefits such as product development, product commercialisation/patent registration or impact on source and society.

This project is on the fundamental research. This research will strengthen the fundamental research done in the country and in particular at USM.

* Sila berikan salinan/*Kindly provide copies*

(c) **Latihan Sumber Manusia**
Training in Human Resources

i) Pelajar Sarjana: –
Graduates Students
(Perincikan nama, ijazah dan status)
(Provide names, degrees and status)

ii) Lain-lain: –
Others

9. **Peralatan yang Telah Dibeli:**
Equipment that has been purchased

1) Buku berjudul 'Chromatic Polynomials and Chromaticity of Graphs', F.M. Dong, K.M. Koh and K.L. Teo, World Scientific 2005.

Tandatangan Penyelidik
Signature of Researcher

5/3/2009
Tarikh

Komen Jawatankuasa Penyelidikan Pusat Pengajian/Pusat
Comments by the Research Committees of Schools/Centres

Prestasi penyelidikan adalah baik

TANDATANGAN PENGERUSI
JAWATANKUASA PENYELIDIKAN
PUSAT PENGAJIAN/PUSAT
Signature of Chairman
[Research Committee of School/Centre]

PROF. MADYA AHMAD IZANI MD. ISMAIL
DEKAN
PUSAT PENGAJIAN SAINS MATEMATIK
UNIVERSITI SAINS MALAYSIA
11800 PULAU PINANG

6/4/09
Tarikh
Date

6. Abstrak Penyelidikan

6.1. Bahasa Inggeris

For the purpose of tackling the four-colour problem, Birkhoff (1912) introduced the chromatic polynomial of a map, denoted by $P(M,\lambda)$, which is a number of proper λ-colouring of a map M. Whitney (1932), who established many fundamental results for it, later generalized the notion of a chromatic polynomial to that of an arbitrary graph. In 1968, Read asked whether it is possible to find a set of necessary and sufficient algebraic conditions for a polynomial to be the chromatic polynomial of some graph. In particular, Read asked for a necessary and sufficient condition for two graphs to be chromatically equivalent; that is, to have the same chromatic polynomial. In 1978, Chao and Whitehead defined a graph to be chromatically unique if no other graphs share its chromatic polynomial. Since then many researchers have been studying chromatic uniqueness and chromatic equivalence of graphs. The question on chromatic equivalence and uniqueness is called the chromaticity problem of graphs. Very recently, Dong, Koh and Teo (2005) finished a monograph on chromatic polynomials and chromaticity of graphs. Salzberg et al. (1985), Tomescu (1987), Peng (1991) and many other reseachers, did the study for chromaticity of complete bipartite graphs.

Dong, Koh and Teo proved in three papers around 1990 that every complete bipartite graph $K(p,q)$, where $p \geq q \geq 2$, is chromatically unique. Since then, the chromaticity of graphs obtained from $K(p,q)$ by removing a set of edges has been studied by several researchers. In 2000 and 2001, Dong, Koh, Teo, Little and Hendy made full use of the notion $\alpha(G,3)$ of the number of 3-independent partitions of the vertex set of a graph G to produce many families of chromatically unique bipartite graphs. In 2005, Roslan and Peng extend some of their main results further. The aim of this project is to study the chromatically unique bipartite graphs with certain edges deleted and in general, to enhance the study of families of chromatically unique graphs.

6.2. Bahasa Malaysia

Bertujuan untuk menyelesaikan masalah Empat-Warna, Birkhoff (1912) memperkenalkan kromatik polinomial bagi suatu peta, ditandakan sebagai $P(M,\lambda)$, iaitu suatu nombor λ-pewarnaan wajar bagi suatu peta M. Whitney (1932), yang menemui banyak keputusan asasi baginya, kemudian telah mengitlakkan kefahaman kromatik polinomial kepada sebarang graf. Pada tahun 1968, Read bertanya samada mungkin mencari suatu set bagi syarat cukup dan perlu (algebra) untuk suatu polinomial menjadi kromatik polinomial graf lain. Secara khusus, Read bertanya tentang syarat cukup dan perlu bagi dua graf untuk menjadi kromatik setara, iaitu, mempunyai kromatik polinomial yang sama. Pada tahun 1978, Chao dan Whitehead menakrifkan suatu graf adalah kromatik unik jika tiada graf lain berkongsi kromatik polinomial dengannya. Sejak itu, ramai penyelidik telah mengkaji keunikan kromatik dan kesetaraan kromatik graf. Persoalan kesetaraan dan keunikan kromatik disebut sebagai masalah kekromatikan graf. Baru-baru ini, Dong, Koh dan Teo (2005) menyiapkan monograf tentang kromatik

polinomial dan kekromatikan graf. Salzberg dan lain-lain (1985), Tomescu (1987), Peng (1991) dan ramai penyelidik lain, telah mengkaji kekromatikan graf bipartit.

Dong, Koh and Teo membuktikan dalam tiga kertas kerja mereka sekitar 1990 bahawa setiap graf bipartit lengkap $K(p,q)$, dengan $p \geq q \geq 2$, adalah kromatik unik. Kemudian, kekromatikan graf diperoleh daripada $K(p,q)$ dengan menyingkirkan suatu set sisi telah dikaji oleh beberapa penyelidik. Pada tahun 2000 dan 2001, Dong, Koh, Little dan Hendy menggunakan sepenuhnya kefahaman $\alpha(G,3)$, iaitu nombor partisi 3-tak bersandar bagi set bucu graf G untuk menghasilkan banyak graf bipartit yang kromatik unik. Pada tahun 2005, Roslan dan Peng telah memperluaskan keputusan penting mereka. Tujuan projek ini ialah untuk mengkaji keunikan kromatik graf bipartit dengan sisi tertentu disingkirkan dan secara amnya, untuk memperkukuhkan lagi kajian famili graf yang unik kromatik.

LAPORAN TEKNIK

PROJEK PENYELIDIKAN JANGKA PENDEK

UNIVERSITI SAINS MALAYSIA

An Attempt to Classify Bipartite Graphs by Their Chromatic Polynomials

No. Akaun: 304/PMATHS/637053

Tempoh: 15 Julai 2006 - 14 Julai 2008

Jumlah: RM 14450.00

Ketua Projek: Dr. Roslan Hasni

ABSTRAK

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ABSTRACT

For the purpose of tackling the four-colour problem, Birkhoff (1912) introduced the chromatic polynomial of a map, denoted by $P(M, \lambda)$, which is a number of proper λ-colouring of a map M. Whitney (1932), who established many fundamental results for it, later generalized the notion of a chromatic polynomial to that of an arbitrary graph. In 1968, Read asked whether it is possible

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Dong, Koh and Teo proved in three papers around 1990 that every complete bipartite graph $K(p,q)$, where $p \geq q \geq 2$, is chromatically unique. Since then, the chromaticity of graphs obtained from K(p,q) by removing a set of edges has been studied by several researchers. In 2000 and 2001, Dong, Koh, Teo, Little and Hendy made full use of the notion $\alpha(G,3)$ of the number of 3-independent partitions of the vertex set of a graph G to produce many families of chromatically unique bipartite graphs. In 2005, Roslan and Peng extend some of their main results further. The aim of this project is to study the chromatically unique bipartite graphs with certain edges deleted and in general, to enhance the study of families of chromatically unique graphs.

INTRODUCTION

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, maximum degree and the chromatic polynomial of G, respectively. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$*equivalent*), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$*it unique*) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint*, or simply $\chi-$*disjoint*.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2$-$connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.

Teo and Koh [11] showed that every graph in $\mathcal{K}(p,q) \cup \mathcal{K}^{-1}(p,q)$ is χ−unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p-q$ is *small enough*, then each graph in $\mathcal{K}^{-s}(p,q)$ is χ−unique;

(ii) If $G \in \mathcal{K}^{-s}(p,q)$, where $0 \leq p-q \leq 1$, such that the set of s edges deleted forms a matching, then G is χ−unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p,q)$, where $3 \leq s \leq p-q$ and

$$q \geq \max\left\{\frac{1}{2}(p-q)(s-1)+\frac{3}{2}, \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6\right\},$$

and the set of s edges deleted forms a matching or a star, then G is χ−unique. In [5], Dong et al. proved that any 2−connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q-1$ or that induces a star is chromatically unique. Very recently, Dong et al. [4] showed that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q-1\}$.

In this project, we continue to study the above problem by investigating three cases:

(1) The chromaticity of $K_2^{-s}(p,q)$ when $p > q = 4$ and $s = 5$,

(2) The chromaticity of $K_2^{-s}(p,q)$ when $p > q = 5$ and $s = 5$,

(3) The chromaticity of $K_2^{-s}(p,q)$ when $p > q = 4$ and $s = 6$.

METHODOLOGY

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [8]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 1: If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G, 3) = \alpha(G, 3) - (2^{|A|-1} + 2^{|B|-1} - 2).$$

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{ Q \mid Q \text{ is an independent sets in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \}.$$

Lemma 2: For $G \in \mathcal{K}^{-s}(p, q)$,

$$\alpha'(G, 3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4−independent partitions $\{ A_1, A_2, A_3, A_4 \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned}(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \\ = (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.\end{aligned}$$

Define

$$\alpha'(G, 4) = \alpha(G, 4) - \{ (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2 \}.$$

Observe that for G, $H \in \mathcal{K}^{-s}(p, q)$,

$$\alpha(G, 4) = \alpha(H, 4) \quad \text{iff} \quad \alpha'(G, 4) = \alpha'(H, 4).$$

The following results will be used to prove our main theorems.

Lemma 3: For $G = (A, B; E) \in \mathcal{K}^{-s}(p, q)$ with $|A| = p$ and $|B| = q$,

$$\begin{aligned}\alpha'(G, 4) \quad = \quad & \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) + \\ & \left| \{ \{ Q_1, Q_2 \} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \} \right|.\end{aligned}$$

Lemma 4: For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1).$$

Theorem 1[3]: For integers p,q,s with $p \geq q \geq 3$ and $0 \leq s \leq 2q-3$, and $G \in \mathcal{K}_2^{-s}(p,q)$,

$$< G > \subseteq \mathcal{K}_2^{-s}(p,q),$$

if one of the following conditions is satisfied:

(i) $s \leq q-1$; (ii) $s = q \geq 6$ and $p \geq 2$; (iii) $p \geq q+4$;

(iv) $p \in \{q+3, q+1\}$ and $0 \leq s \leq 2q-4$;

(v) $p = q+2$ and $\Delta(G') \geq s+3-q$;

(vi) $p = q$ and $\alpha'(G_i, 3) < 2^{p-2}$.

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9. C.P. Teo, K.M. Koh, The chromaticity of complete bipartite graphs with at most one edge deleted, *J. Graph Theory* **14** (1990), 89–99.

Roslan Hasni

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A Family Of Chromatically Unique Bipartite Graph

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ABSTRACT

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of 2−connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. F.M.Dong et al. (Discrete Math. vol.224 (2000) 107 124) proved that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq min\{4, q-1\}$, then G is chromatically unique. In [10], we extended this result to $p > q = 5$ and $s = 5$. In this paper, we study the chromaticity of any graph $G \in \mathcal{K}_2^{-s}(p,q)$ when $p > q = 4$ and $s = 5$.

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1 Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be *chromatically equivalent* (or simply χ−equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ−unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ*−closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply χ*−disjoint*.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2−connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.

In [5], Dong et al. proved the following result.

Theorem 1.1 *For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q-1$, $K_2^{-s}(p,q)$ is χ-closed.*

Teo and Koh [10] showed that every graph in $\mathcal{K}(p,q) \cup \mathcal{K}^{-1}(p,q)$ is χ−unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p-q$ is *small enough*, then each graph in $\mathcal{K}^{-s}(p,q)$ is χ−unique;

(ii) If $G \in \mathcal{K}^{-s}(p,q)$, where $0 \leq p-q \leq 1$, such that the set of s edges deleted forms a matching, then G is χ−unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p,q)$, where $3 \leq s \leq p-q$ and

$$q \geq \max\left\{\frac{1}{2}(p-q)(s-1)+\frac{3}{2}, \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6\right\},$$

and the set of s edges deleted forms a matching or a star, then G is χ−unique. In [5], Dong et al. proved that any 2−connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q-1$ or that induces a star is chromatically unique.

Very recently, Dong et al. [4] showed that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q-1\}$. In [9], we extend this result to $p \geq q \geq 6$ and $s = 5$ or 6. In [10], we proved for the case $p > q = 5$ and $s = 5$. In this paper, we shall study the chromaticity of $K_2^{-s}(p,q)$ when $p > q = 4$ and $s = 5$.

2 Preliminary results and notation

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a k-*independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [8]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 2.1 *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{1}$$

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{\, Q \mid Q \text{ is an independent set in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \,\}.$$

Lemma 2.2 *(Dong et al. [5]) For $G \in \mathcal{K}^{-s}(p,q)$,*

$$\alpha'(G,3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4–independent partitions $\{ A_1, A_2, A_3, A_4 \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned}
&(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \\
&= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.
\end{aligned}$$

Define

$$\alpha'(G,4) = \alpha(G,4) - \{\, (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2 \,\}.$$

Observe that for G, $H \in \mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4) = \alpha(H,4) \quad \text{if and only if} \quad \alpha'(G,4) = \alpha'(H,4).$$

The following results will be used to prove our main theorem.

Lemma 2.3 *(Dong et al. [3]) For $G = (A, B; E) \in \mathcal{K}^{-s}(p,q)$ with $|A| = p$ and $|B| = q$,*

$$\begin{aligned}
\alpha'(G,4) \;&=\; \sum_{Q \in \Omega(G)} (2^{p-1-|Q\cap A|} + 2^{q-1-|Q\cap B|} - 2) + \\
&\quad \left| \{\, \{ Q_1, Q_2 \} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \,\} \right|.
\end{aligned}$$

Theorem 2.1 *(Dong et al. [5]) For integers p, q, s with $p \geq q \geq 3$ and $0 \leq s \leq 2q - 3$, and $G \in \mathcal{K}_2^{-s}(p,q)$,*

$$[G] \subseteq \mathcal{K}_2^{-s}(p,q),$$

if one of the following conditions is satisfied:

(i) $s \leq q-1$;

(ii) $s=q \geq 6$ and $p \geq 2$;

(iii) $p \geq q+4$;

(iv) $p \in \{q+3, q+1\}$ and $0 \leq s \leq 2q-4$;

(v) $p=q+2$ and $\Delta(G') \geq s+3-q$;

(vi) $p=q$ and $\alpha'(G_i,3) < 2^{p-2}$.

Theorem 2.2 *(Peng [7]) If H is $\chi-$equivalent to G, and G is a bipartite graph, then H is also a bipartite graph having the same number of vertices, edges, pure quadrilaterals, and complete subgraphs $K(2,3)$ as G.*

3 Main result

In [9], we proved that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 6$ and $s=5$ or $s=6$. In [10], we showed that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq q=5$ and $s=5$. In this section, we shall show that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq q=4$ and $s=5$.

Let G be any graph in $\mathcal{K}_2^{-5}(p,q)$, and $G'=K(p,q)-G$. There are 21 structures of G' ($q=4$ and G is 2−connected), which are named as G_1', G_2', …, G_{21}' (see Table 1). We group the graphs G_1, G_2, …, G_{21} according to their values of $\alpha'(G_i,3)$, which can be calculated by using Lemma 2.2 and these values are in column three of Table 1. Thus we have the following observations.

(i) $\alpha'(G_i,3)=6$, for i=1;

(ii) $\alpha'(G_i,3)=7$, for i=2,3,4;

(iii) $\alpha'(G_i,3)=8$, for i=5,6,7,8;

(iv) $\alpha'(G_i,3)=9$, for i=9,10;

(v) $\alpha'(G_i,3)=10$, for i=11,12,13,14;

(vi) $\alpha'(G_i,3)=11$, for i=15,16,17;

(vii) $\alpha'(G_i,3)=13$, for i=18;

(viii) $\alpha'(G_i, 3) = 16$, for i=19;

(ix) $\alpha'(G_i, 3) = 17$, for i=20;

(x) $\alpha'(G_i, 3) = 31$, for i=21.

We then group these graphs according to their $\alpha'(G_i, 3)$. Hence we have the following classification of the graphs.

$$
\begin{aligned}
\mathcal{T}_1 &= \{\, G_1 \,\} \\
\mathcal{T}_2 &= \{\, G_2, G_3, G_4 \,\} \\
\mathcal{T}_3 &= \{\, G_5, G_6, G_7, G_8 \,\} \\
\mathcal{T}_4 &= \{\, G_9, G_{10} \,\} \\
\mathcal{T}_5 &= \{\, G_{11}, G_{12}, G_{13}, G_{14} \,\} \\
\mathcal{T}_6 &= \{\, G_{15}, G_{16}, G_{17} \,\} \\
\mathcal{T}_7 &= \{\, G_{18} \,\} \\
\mathcal{T}_8 &= \{\, G_{19} \,\} \\
\mathcal{T}_9 &= \{\, G_{20} \,\} \\
\mathcal{T}_{10} &= \{\, G_{21} \,\}
\end{aligned}
$$

We also calculate the values of $\alpha'(G_i, 4)$ by using Lemma 2.3 and we list them in column four of Table 1.

We now present our main result in the following theorem.

Theorem 3.1 *Every graph in $\mathcal{K}_2^{-5}(p,q)$ with $p > q = 4$ is χ-unique if one of the following conditions is satisfied:* p 7, 5 ✓

(i) $p \geq 8$,

(ii) $p = 7$ *or* 5,

(iii) $p = 6$ *and* $\Delta(G') \geq 4$.

Proof Observe that for any i, j with $1 \leq i < j \leq 10$, $\alpha'(G,3) < \alpha'(H,3)$ if $G \in \mathcal{T}_i$ and $H \in \mathcal{T}_j$. Thus by Lemma 2.1 and Equation (1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 10)$ are $\chi-$disjoint and since $\mathcal{K}_2^{-5}(p,4)$ is $\chi-$closed under the conditions (i), (ii) or (iii) (see Theorem 2.1), then each $\mathcal{T}_i$ $(1 \leq i \leq 10)$ is $\chi-$closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are $\chi-$unique, it suffices to show that for any two graphs, G, $H \in \mathcal{T}_i$, if $G \not\cong H$, then either $\alpha'(G,4) \neq \alpha'(H,4)$ or by using Theorem 2.2. Note that $\mathcal{T}_1$, $\mathcal{T}_7$, $\mathcal{T}_8$, $\mathcal{T}_9$ and $\mathcal{T}_{10}$ contains only one graph G_1, G_{18},G_{19}, G_{20} and G_{21}, respectively and hence G_1, G_{18},G_{19}, G_{20} and G_{21} are $\chi-$unique. The remaining work is to compare every two graphs in $\mathcal{T}_i$ for $2 \leq i \leq 6$.

(1) $\mathcal{T}_2$: We consider two cases.

(1.1) Case 1: $p = 5$. Notice that the graph G_2 is not considerable since $p = 5$ and thus we only consider graphs G_3 and G_4.

$$\begin{aligned}
&\alpha'(G_4,4) - \alpha'(G_3,4) \\
&= \left[\, 3 \cdot 2^{p-3} + 3 \cdot 2^{q-3} + 8 \,\right] - \left[\, 3 \cdot 2^{p-3} + 3 \cdot 2^{q-3} + 11 \,\right] \\
&= -3 < 0.
\end{aligned}$$

Thus, we have $\alpha'(G_4,4) < \alpha'(G_3,4)$.

(1.2) Case 2: $p \geq 7$.

$$\begin{aligned}
&\alpha'(G_2,4) - \alpha'(G_4,4) \\
&= \left[\, 2 \cdot 2^{p-3} + 2 \cdot 2^{q-2} + 11 \,\right] - \left[\, 3 \cdot 2^{p-3} + 3 \cdot 2^{q-3} + 8 \,\right] \\
&= -2^{p-3} + 2^{q-3} + 3 < 0,
\end{aligned}$$

$$\alpha'(G_4,4) - \alpha'(G_3,4) = -3 < 0.$$

Thus, we have $\alpha'(G_2,4) < \alpha'(G_4,4) < \alpha'(G_3,4)$.

(2) $\mathcal{T}_3$: We consider two cases.

(2.1) Case 1: $p = 5$.

$$\begin{aligned}
&\alpha'(G_5, 4) - \alpha'(G_6, 4)\\
&= \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 7\right] - \left[4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 7\right]\\
&= -2^{p-3} + 2^{q-3} < 0,
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_6, 4) - \alpha'(G_7, 4)\\
&= \left[4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 7\right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 10\right]\\
&= 2^{p-3} - 2^{q-3} - 3 < 0,
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_7, 4) - \alpha'(G_8, 4)\\
&= \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 10\right] - \left[4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 10\right]\\
&= -2^{p-3} + 2^{q-3} < 0.
\end{aligned}$$

Thus, we have $\alpha'(G_5, 4) < \alpha'(G_6, 4) < \alpha'(G_7, 4) < \alpha'(G_8, 4)$.

(2.2) Case 2: $p \geq 7$.

$$\begin{aligned}
&\alpha'(G_5, 4) - \alpha'(G_7, 4)\\
&= \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 7\right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 10\right]\\
&= -3 < 0,
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_7, 4) - \alpha'(G_6, 4)\\
&= -2^{p-3} + 2^{q-3} + 3 < 0,
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_6, 4) - \alpha'(G_8, 4)\\
&= \left[4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 7\right] - \left[4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 10\right]\\
&= -3 < 0.
\end{aligned}$$

Thus, we have $\alpha'(G_5, 4) < \alpha'(G_7, 4) < \alpha'(G_6, 4) < \alpha'(G_8, 4)$.

(3) $\mathcal{T}_4$.

$$\alpha'(G_9, 4) - \alpha'(G_{10}, 4)$$

$$= \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 5 \right] - \left[7 \cdot 2^{p-4} + 4 \cdot 2^{q-2} + 7 \right]$$
$$= 5 \cdot 2^{p-4} - 2 \cdot 2^{q-3} - 2 > 0.$$

Thus, we have $\alpha'(G_{10}) < \alpha'(G_9, 4)$.

(4) $\mathcal{T}_5$: We consider two cases.

(4.1) Case 1: $p = 5$.

$$\alpha'(G_{12}, 4) - \alpha'(G_{11}, 4)$$
$$= \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 2 \right] - \left[7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 1 \right]$$
$$= -3 \cdot 2^{p-4} + 2 \cdot 2^{q-3} + 1 < 0,$$

$$\alpha'(G_{11}, 4) - \alpha'(G_{13}, 4)$$
$$= \left[7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 1 \right] - \left[9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 11 \right]$$
$$= 5 \cdot 2^{p-4} - 3 \cdot 2^{q-3} - 10 < 0,$$

$$\alpha'(G_{13}, 4) - \alpha'(G_{14}, 4)$$
$$= \left[9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 11 \right] - \left[9 \cdot 2^{q-3} + 11 \cdot 2^{p-4} + 11 \right]$$
$$= -2^{p-3} + 2^{q-3}.$$

Thus, we have $\alpha'(G_{12}, 4) < \alpha'(G_{11}, 4) < \alpha'(G_{13}, 4) < \alpha'(G_{14}, 4)$.

(4.2) Case 2: $p \geq 7$.

$$\alpha'(G_{13}, 4) - \alpha'(G_{12}, 4)$$
$$= \left[9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 11 \right] - \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 2 \right]$$
$$= -2 \cdot 2^{p-4} + 2^{q-3} + 9 < 0,$$

$$\alpha'(G_{12}, 4) - \alpha'(G_{14}, 4)$$
$$= \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 2 \right] - \left[9 \cdot 2^{q-3} + 11 \cdot 2^{p-4} + 11 \right]$$
$$= -9 < 0,$$

$$\alpha'(G_{14}, 4) - \alpha'(G_{11}, 4)$$
$$= \left[9 \cdot 2^{q-3} + 11 \cdot 2^{p-4} + 11 \right] - \left[7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 1 \right]$$

$$= -3 \cdot 2^{p-4} + 2 \cdot 2^{q-3} + 10 < 0.$$

Thus, we have $\alpha'(G_{13}, 4) < \alpha'(G_{12}, 4) < \alpha'(G_{14}, 4) < \alpha'(G_{11}, 4)$.

(5) $\mathcal{T}_6$: We consider two cases.

(5.1) **Case 1:** $p = 5$.

$$\begin{aligned}
&\alpha'(G_{15}, 4) - \alpha'(G_{17}, 4) \\
&= \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} - 3 \right] - \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 1 \right] \\
&= 2 \cdot 2^{p-4} - 2^{q-3} - 4 < 0,
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_{17}, 4) - \alpha'(G_{16}, 4) \\
&= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 1 \right] - \left[15 \cdot 2^{q-4} + 10 \cdot 2^{p-3} - 3 \right] \\
&= -7 \cdot 2^{p-4} + 7 \cdot 2^{q-4} + 4 < 0.
\end{aligned}$$

Thus, we have $\alpha'(G_{15}, 4) < \alpha'(G_{17}, 4) < \alpha'(G_{16}, 4)$.

(5.2) **Case 2:** $p \geq 7$.

$$\begin{aligned}
&\alpha'(G_{17}, 4) - \alpha'(G_{15}, 4) \\
&= -2 \cdot 2^{p-4} + 2^{q-3} + 4 < 0,
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_{15}, 4) - \alpha'(G_{16}, 4) \\
&= \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} - 3 \right] - \left[15 \cdot 2^{q-4} + 10 \cdot 2^{p-3} - 3 \right] \\
&= -5 \cdot 2^{p-4} + 5 \cdot 2^{q-4} < 0.
\end{aligned}$$

Thus, we have $\alpha'(G_{17}, 4) < \alpha'(G_{15}, 4) < \alpha'(G_{16}, 4)$.

Therefore, by (1) – (5), we have the following observation:

The theorem holds under the conditions (i) or (ii). But for condition (iii), we only need to consider the graphs in $\mathcal{T}_i$ $(8 \leq i \leq 10)$ since $\Delta(G') \geq 4$ for G_{19}, G_{20} and G_{21} and it is clear that these graphs are χ-unique. Also notice that these graphs are not considerable for case $p = 5$. Hence by the detail comparisons above (for graphs in $\mathcal{T}_i$ $(2 \leq i \leq 6)$), we conclude that theorem holds under conditions (i), (ii) and (iii). This completes the proof of Theorem 3.1. $\square$

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TABLE 1 (1 of 3)

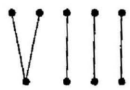
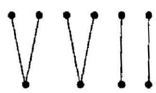
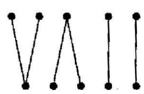
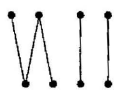
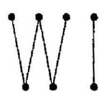
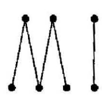
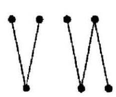
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,q} - G_i$) $\lvert A'\rvert \le p,\ \lvert B'\rvert \le q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 5(2^{p-2} + 2^{q-2} - 2)$
G_1	 A' B'	6	$(2^{p-3} + 2^{q-2} - 2) + 12$
G_2	 A' B'	7	$2(2^{p-3} + 2^{q-2} - 2) + 15$
G_3	 A' B'	7	$3 \cdot 2^{p-3} + 3 \cdot 2^{q-3} + 11$
G_4	 A' B'	7	$3 \cdot 2^{p-3} + 3 \cdot 2^{q-3} + 8$
G_5	 A' B'	8	$4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 7$
G_6	 A' B'	8	$4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 7$
G_7	 A' B'	8	$4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 10$
G_8	 A' B'	8	$4 \cdot 2^{q-3} + 5 \cdot 2^{p-3} + 10$

TABLE 1 (2 of 3)

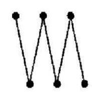
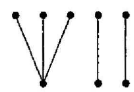
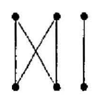
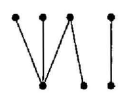
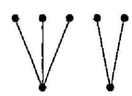
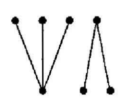
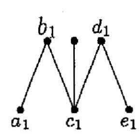
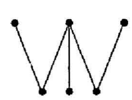
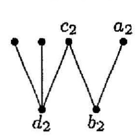
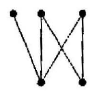
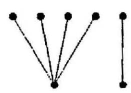
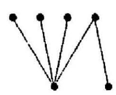
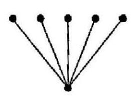
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,q} - G_i$) $\lvert A'\rvert \leq p,\ \lvert B'\rvert \leq q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 5(2^{p-2} + 2^{q-2} - 2)$
G_9	 A' B'	9	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 5$
G_{10}	 A' B'	9	$7 \cdot 2^{p-4} + 4 \cdot 2^{q-2} + 7$
G_{11}	 A' B'	10	$7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 1$
G_{12}	 A' B'	10	$11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 2$
G_{13}	 A' B'	10	$9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 11$
G_{14}	 A' B'	10	$9 \cdot 2^{q-3} + 11 \cdot 2^{p-4} + 11$
G_{15}	 A' B'	11	$15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} - 3$
G_{16}	 A' B'	11	$15 \cdot 2^{q-4} + 10 \cdot 2^{p-3} - 3$
G_{17}	 A' B'	11	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 1$

TABLE 1 (3 of 3)

Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,q} - G_i$) $\lvert A'\rvert \le p$, $\lvert B'\rvert \le q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 5(2^{p-2} + 2^{q-2} - 2)$
G_{18}	 A' B'	13	$19 \cdot 2^{p-4} + 13 \cdot 2^{q-3} - 9$
G_{19}	 A' B'	16	$16 \cdot 2^{p-4} + 2^{p-5} + 11 \cdot 2^{q-2} - 7$
G_{20}	 A' B'	17	$9 \cdot 2^{p-5} + 4 \cdot 2^{p-2} + 2^{q-3} + 11 \cdot 2^{q-2} - 17$
G_{21}	 A' B'	31	$15 \cdot 2^{p-3} + 11 \cdot 2^{p-6} + 26 \cdot 2^{q-2} - 52$

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CHROMATICITY OF CERTAIN BIPARTITE GRAPH WITH FIVE EDGES DELETED

Roslan Hasni & Y.H. Peng

ABSTRACT: For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of 2-connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. F.M. Dong et al. (Discrete Math. vol. 224 (2000) 107–124) proved that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq \min\{4, q-1\}$, then G is chromatically unique. In [9], we extended this result to $p \geq q \geq 6$ and $s = 5$ or $s = 6$. In this paper, we discuss the chromaticity of any graph $G \in \mathcal{K}_2^{-s}(p,q)$ when $p \geq q = 5$ and $s = 5$.

***Keywords:** Chromatic polynomial, Chromatically equivalence, Chromatically unique.*

1. INTRODUCTION

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $\Delta(G)$ and $P(G, \lambda)$ be the vertex set, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be chromatically equivalent (or simply χ–equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is chromatically unique (or simply χ-unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ-closed. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1, \lambda) \neq P(G_2, \lambda)$ for every $G_1 \in \mathcal{G}_1$ and $\mathcal{G}_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply χ-disjoint.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p, q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$ denote the set of connected (resp. 2–connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. In [5], Dong *et al.* proved the following result.

Theorem 1.1: For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q - 1$, $\mathcal{K}_2^{-s}(p,q)$ is χ-closed.

Teo and Koh [11] showed that every graph in $\mathcal{K}(p, q) \cup \mathcal{K}^{-1}(p, q)$ is χ-unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \le s \le 4$ and $p - q$ is small enough, then each graph in $\mathcal{K}^{-s}(p, q)$ is χ-unique;

(ii) If $G \in \mathcal{K}^{-s}(p, q)$, where $0 \le p - q \le 1$, such that the set of s edges deleted forms a matching, then G is χ-unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p, q)$, where $3 \le s \le p - q$ and

$$q \ge \max\left\{\frac{1}{2}(p-q)(s-1)+\frac{3}{2}, \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6\right\},$$

and the set of s edges deleted forms a matching or a star, then G is χ-unique. In [5], Dong *et al.* proved that any 2-connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q - 1$ or that induces a star is chromatically unique.

Very recently, Dong *et al.* [4] showed that any graph in $\mathcal{K}_2^{-s}(p,q)$ χ-unique if $p \ge q \ge 3$ and $1 \le s \le \min\{4, q - 1\}$. In [9], we extend this result to $p \ge q \ge 6$ and $s = 5$ or 6. In this paper, we shall present the case when $p \ge q = 5$ and $s = 5$.

2. PRELIMINARY RESULTS AND NOTATION

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \notin x \in y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [8]):

$$P(G,\lambda)=\sum_{k=1}^{n}\alpha(G,k)\lambda(\lambda-1)\ldots(\lambda-k+1)$$

Thus, we have

Lemma 2.1: If $G \sim H$, then $\alpha(G, k) = \alpha(H, k)$ for $k = 1, 2, \ldots$.

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G, 3) = \alpha(G, 3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{1}$$

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{Q \mid Q \text{ is an independent sets in } G \text{ with } Q \cap A \neq \theta, Q \cap B \neq\}.$$

Lemma 2.2: (Dong *et al.* [5]) For $G \in \mathcal{K}^{-s}(p, q)$,

$$\alpha'(G, 3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4-independent partitions $\{A_1, A_2, A_3, A_4\}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$(2^{|A|-1}-1)(2^{|B|-1}-1)+\frac{1}{3!}(3^{|A|}-3.2^{|A|}+3)+\frac{1}{3!}(3^{|B|}-3\cdot 2^{|B|}+3)$$

$$=(2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2$$

Define

$$\alpha'(G,4)=\alpha(G,4)-\{2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2\}.$$

Observe that for $G, H \in \mathcal{K}^{-s}(p, q)$,

$$\alpha(G, 4) = \alpha(H, 4) \text{ iff } \alpha'(G, 4) = \alpha'(H, 4).$$

The following results will be used to prove our main theorems.

Lemma 2.3: (Dong *et al.* [3]) For $G = (A, B; E) \in \mathcal{K}^{-s}(p, q)$ with $|A| = p$ and $|B| = q$,

$$\alpha'(G,4) = \sum_{Q\in\Omega(G)} (2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)+$$

$$|\{\{Q_1,Q_2\}|Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\theta\}|.$$

Lemma 2.4: (Dong *et al.* [5]) For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1).$$

Theorem 2.1: (Dong *et al.* [5]) For integers p, q, s with $p \leq q \geq 3$ and $0 \leq s \leq 2q - 3$, and $G \in \mathcal{K}_2^{-s}(p,q)$

$$<G> \subseteq \mathcal{K}_2^{-s}(p,q)$$

if one of the following conditions is satisfied:

(i) $s \leq q - 1$;

(ii) $s = q \geq 6$ and $p \geq 2$;

(iii) $p \geq q + 4$;

(iv) $p \in \{q + 3, q + 1\}$ and $0 \leq s \leq 2q - 4$;

(v) $p = q + 2$ and $\Delta(G') \geq s + 3 - q$;

(vi) $p = q$ and $\alpha'(G_i, 3) < 2^{p-2}$.

3. MAIN RESULT

In [9], we proved that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 6$ and $s = 5$ or $s = 6$. In this section, we shall show that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq q = 5$ and $s = 5$.

Let G be any graph in $\mathcal{K}_2^{-5}(p,q)$ and $G' = K(p, q) - G$. There are 34 structures of G', which are named as $G'_1, G'_2, \ldots, G'_{34}$ (see Table 1 in [10]). We group the graphs $G_1, G_2, \ldots, G_{34}$ according to their values of $\alpha'(G_i, 3)$, which can be calculated by using Lemma 2.2 and these values are in column three of Table 1 [10]. Thus we have the following observations.

(i) $\alpha'(G_i, 3) = 5$, for $i = 1$;

(ii) $\alpha'(G_i, 3) = 6$, for $i = 2,3$;

(iii) $\alpha'(G_i, 3) = 7$, for $i = 4, 5, 6, 7$;

(iv) $\alpha'(G_i, 3) = 8$, for $i = 8, 9, 10, 11$;

(v) $\alpha'(G_i, 3) = 9$, for $i = 12, 13, 14$;

(vi) $\alpha'(G_i, 3) = 10$, for $i = 15, 16, \ldots, 21$;

(vii) $\alpha'(G_i, 3) = 11$, for $i = 22, 23, 24, 25$;

(viii) $\alpha'(G_i, 3) = 13$, for $i = 26, 27, 28$;

(ix) $\alpha'(G_i, 3) = 16$, for $i = 29, 30$;

(x) $\alpha'(G_i, 3) = 17$, for $i = 31, 32$;

(xi) $\alpha'(G_i, 3) = 31$, for $i = 33, 34$.

We then group these graphs according to their $\alpha'(G_i, 3)$. Hence we have the following classification of the graphs.

$$\mathcal{T}_1 = \{G_1\}$$
$$\mathcal{T}_2 = \{G_2, G_3\}$$
$$\mathcal{T}_3 = \{G_4, G_5, G_6, G_7\}$$
$$\mathcal{T}_4 = \{G_8, G_9, G_{10}, G_{11}\}$$

$$\mathcal{T}_5 = \{G_{12}, G_{13}, G_{14}\}$$
$$\mathcal{T}_6 = \{G_{15}, G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}\}$$
$$\mathcal{T}_7 = \{G_{22}, G_{23}, G_{24}, G_{25}\}$$
$$\mathcal{T}_8 = \{G_{26}, G_{27}, G_{28}\}$$
$$\mathcal{T}_9 = \{G_{29}, G_{30}\}$$
$$\mathcal{T}_{10} = \{G_{31}, G_{32}\}$$
$$\mathcal{T}_{11} = \{G_{33}, G_{34}\}$$

We also calculate the values of $\alpha'(G_i, 4)$ by using Lemma 2.3 and we list them in column four of Table 1 [10].

We now present our main result in the following theorem.

Theorem 3.1: Every graph in $\mathcal{K}_2^{-5}(p,q)$ with $p > q = 5$ is χ-unique if one of the following conditions is satisfied:

(i) $p \geq 9$,

(ii) $p = 6$ or 8,

(iii) $p = 7$ and $\Delta\,(G') \geq 3$.

Proof: Observe that for any i, j with $1 \leq i < j \leq 11$, $\alpha'(G, 3) < \alpha'(H, 3)$ if $G \in \mathcal{T}_i$ and $H \in \mathcal{T}_j$. Thus by Lemma 2.1 and Equation (1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 11)$ are χ-disjoint and since $\mathcal{K}_2^{-5}(p,5)$ is χ-closed under the conditions (i), (ii) or (iii) (see Theorem 2.1), then each $\mathcal{T}_i$ $(1 \leq i \leq 11)$ is χ-closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are -unique, it suffices to show that for any two graphs, $G, H \in \mathcal{T}_i$, if $G \not\cong$ H, then either $\alpha'(G, 4) \neq \alpha'(H, 4)$ or $\alpha(G, 5) \neq (H, 5)$. Note that $\mathcal{T}_1$ contain only one graph G_1, and hence G_1 is χ-unique. The remaining work is to compare every two graphs in $\mathcal{T}_i$ for $2 \leq i \leq 11$. Since the methods used in comparing every two graphs in $\mathcal{T}_i$ is standard, long and rather repetitive, we shall not discuss all here. In the following, we shall establish several inequality of the form $\alpha'(G_i, 4) < \alpha'(G_j, 4)$ for some i, j.

(1) $\mathcal{T}_2$: $\alpha'(G_2, 4) - \alpha'(G_3, 4) < 0$.

(2) $\mathcal{T}_3$: We have

(i) $\alpha'(G_4, 4) - \alpha'(G_7, 4) < 0$,

(ii) $\alpha'(G_7, 4) - \alpha'(G_5, 4) < 0$,

(iii) $\alpha'(G_5, 4) - 0(G_6, 4) < 0$.

Thus, from (i), (ii) and (iii), we can conclude that

$$\alpha'(G_4, 4) < \alpha'(G_7, 4) < \alpha'(G_5, 4) < \alpha'(G_6, 4).$$

(3) $\mathcal{T}_4$: We have

(i) $\alpha'(G_8, 4) - \alpha'(G_{10}, 4) < 0,$

(ii) $\alpha'(G_{10}, 4) - \alpha'(G_9, 4) < 0,$

(iii) $\alpha'(G_9, 4) - \alpha'(G_{11}, 4) < 0.$

Thus, from (i), (ii) and (iii), we can conclude that

$$\alpha'(G_8, 4) < \alpha'(G_{10}, 4) < \alpha'(G_9, 4) < \alpha'(G_{11}, 4).$$

(4) $\mathcal{T}_5$: We can show that

(i) $\alpha'(G_{13}, 4) - \alpha'(G_{12}, 4) < 0,$

(ii) $\alpha'(G_{12}, 4) - \alpha'(G_{14}, 4) < 0.$

Thus, from (i) and (ii), we can conclude that

$$\alpha'(G_{13}, 4) < \alpha'(G_{12}, 4) < \alpha'(G14, 4).$$

For $\mathcal{T}_i$, where $5 \le i \le 11$, we can similarly show that for any two graphs, $G, H \in \mathcal{T}_i$, if $G \not\cong H$, then either $\alpha'(G, 4) \neq \alpha'(H, 4)$ or $(G, 5) \neq \alpha(H, 5)$. For the detail proof, the reader may refer to [10]. This completes the proof of Theorem 3.1.

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Chromatic Uniqueness of Certain Bipartite Graphs With Six Edges Deleted

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ABSTRACT

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of $2-$connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. F.M.Dong et al. (Discrete Math. vol.224 (2000) 107–124) proved that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq \min\ \{4, q-1\}$, then G is chromatically unique. In this paper, we study the chromaticity of any graph $G \in \mathcal{K}_2^{-s}(p,q)$ when $p \geq 6$, $q = 4$ and $s = 6$.

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1 Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed.* For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply $\chi-$*disjoint.*

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. $2-$connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.

In [4,5], Dong et al. proved the following results.

Lemma 1.1 *If $p \geq q \geq 3$ and $s \leq p+q-4$, then for any $G \in K^{-s}(p,q)$ with $\delta(G) \geq 2$, then G is 2-connected.*

Theorem 1.1 *For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q-1$, $K_2^{-s}(p,q)$ is χ-closed.*

Teo and Koh [13] showed that every graph in $\mathcal{K}(p,q) \cup \mathcal{K}^{-1}(p,q)$ is $\chi-$unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p-q$ is *small enough*, then each graph in $\mathcal{K}^{-s}(p,q)$ is $\chi-$unique;

(ii) If $G \in \mathcal{K}^{-s}(p,q)$, where $0 \leq p-q \leq 1$, such that the set of s edges deleted forms a matching, then G is $\chi-$unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p,q)$, where $3 \leq s \leq p-q$ and

$$q \geq \max\left\{\frac{1}{2}(p-q)(s-1)+\frac{3}{2}, \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6\right\},$$

and the set of s edges deleted forms a matching or a star, then G is $\chi-$unique. In [5], Dong et al. proved that any 2$-$connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q-1$ or that induces a star is chromatically unique.

Very recently, Dong et al. [4] showed that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q-1\}$. In [9], we proved that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 6$ and $s=5$; or $p \geq q \geq 7$ and $s=6$. In [10,11], we extended this study for the case $p > q = 5$ and $s=5$; or $p > q = 4$ and $s=5$. In this paper, we shall study the chromaticity of any graph in $K_2^{-s}(p,q)$ when $p \geq 6$, $q=4$ and $s=6$.

2 Preliminary Results and Notation

For a bipartite graph $G=(A,B;E)$ with bipartition A and B and edge set E, let $G'=(A',B';E')$ be the bipartite graph induced by the edge set $E'=\{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G'=K_{p,q}-G$, where $p=|A|$ and $q=|B|$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [8]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 2.1 *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{1}$$

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{\, Q \mid Q \text{ is an independent set in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \,\}.$$

Lemma 2.2 *(Dong et al. [5]) For $G \in \mathcal{K}^{-s}(p,q)$,*

$$\alpha'(G,3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4−independent partitions $\{A_1, A_2, A_3, A_4\}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned}
&(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \\
&= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.
\end{aligned}$$

Define

$$\alpha'(G,4) = \alpha(G,4) - \{\, (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2 \,\}.$$

Observe that for G, $H \in \mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4) = \alpha(H,4) \quad \text{if and only if} \quad \alpha'(G,4) = \alpha'(H,4).$$

The following results will be used to prove our main theorem.

Lemma 2.3 *(Dong et al. [3]) For $G = (A, B; E) \in \mathcal{K}^{-s}(p,q)$ with $|A| = p$ and $|B| = q$,*

$$\begin{aligned}
\alpha'(G,4) \quad = \quad & \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) + \\
& \left| \{\, \{Q_1, Q_2\} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \,\} \right|.
\end{aligned}$$

Lemma 2.4 *(Dong et al. [5]) For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,*

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1).$$

Theorem 2.1 *(Dong et al. [5]) For integers p, q, s with $p \geq q \geq 3$ and $0 \leq s \leq 2q - 3$, and $G \in \mathcal{K}_2^{-s}(p, q)$,*

$$[G] \subseteq \mathcal{K}_2^{-s}(p, q),$$

if one of the following conditions is satisfied:

(i) $s \leq q - 1$;

(ii) $s = q \geq 6$ *and* $p \geq 2$;

(iii) $p \geq q + 4$;

(iv) $p \in \{q + 3, q + 1\}$ *and* $0 \leq s \leq 2q - 4$;

(v) $p = q + 2$ *and* $\Delta(G') \geq s + 3 - q$;

(vi) $p = q$ *and* $\alpha'(G_i, 3) < 2^{p-2}$

3 Main Result

In [9], we proved that every graph in $\mathcal{K}_2^{-s}(p, q)$ is χ-unique if $p \geq q \geq 6$ and $s = 5$ or $s = 6$. In [10], we showed that every graph in $\mathcal{K}_2^{-s}(p, q)$ is χ-unique if $p > q = 5$ and $s = 5$. In [11], we proved that every graph in $\mathcal{K}_2^{-s}(p, q)$ is χ-unique if $p > q = 4$ and $s = 5$. In this section, we shall prove that every graph in $\mathcal{K}_2^{-s}(p, q)$ is χ-unique if $p \geq 6$, $q = 4$ and $s = 6$.

Let G be any graph in $\mathcal{K}_2^{-6}(p, q)$, and $G' = K(p, q) - G$. By construction method and Lemma 1.1, one can easily verify that there are 44 structures of G' ($q = 4$ and G is 2−connected), which are named as G'_1, G'_2, ..., G'_{44} (see Table 1 in [12]). We group the graphs G_1, G_2, ..., G_{44} according to their values of $\alpha'(G_i, 3)$, which can be calculated by using Lemma 2.2 and these values are in column three of Table 1. Thus we have the following observations.

(i) $\alpha'(G_i, 3) = 8$, for i=1;

(ii) $\alpha'(G_i, 3) = 9$, for i=2,3,4,5;

(iii) $\alpha'(G_i, 3) = 10$, for i=6,7,...,11;

(iv) $\alpha'(G_i, 3) = 11$, for i=12,13,...,17;

(v) $\alpha'(G_i, 3) = 12$, for i=18,19,...,25;

(vi) $\alpha'(G_i, 3) = 13$, for i=26,27,28;

(vii) $\alpha'(G_i, 3) = 14$, for i=29,30;

(viii) $\alpha'(G_i, 3) = 15$, for i=31,32;

(ix) $\alpha'(G_i, 3) = 17$, for i=33,34;

(x) $\alpha'(G_i, 3) = 18$, for i=35,36,37;

(xi) $\alpha'(G_i, 3) = 19$, for i=38,39;

(xii) $\alpha'(G_i, 3) = 20$, for i=40;

(xiii) $\alpha'(G_i, 3) = 21$, for i=41;

(xiv) $\alpha'(G_i, 3) = 32$, for i=42;

(xv) $\alpha'(G_i, 3) = 33$, for i=43;

(xvi $\alpha'(G_i, 3) = 63$, for i=44.

We then group these graphs according to their $\alpha'(G_i, 3)$. Hence we have the following classification of the graphs.

$$
\begin{aligned}
\mathcal{T}_1 &= \{ G_1 \} \\
\mathcal{T}_2 &= \{ G_2, G_3, G_4, G_5 \} \\
\mathcal{T}_3 &= \{ G_6, G_7, \ldots, G_{11} \} \\
\mathcal{T}_4 &= \{ G_{12}, G_{13}, \ldots, G_{17} \} \\
\mathcal{T}_5 &= \{ G_{18}, G_{19}, \ldots, G_{25} \} \\
\mathcal{T}_6 &= \{ G_{26}, G_{27}, G_{28} \} \\
\mathcal{T}_7 &= \{ G_{29}, G_{30} \} \\
\mathcal{T}_8 &= \{ G_{31}, G_{32} \}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_9 &= \{ G_{33}, G_{34} \} \\
\mathcal{T}_{10} &= \{ G_{35}, G_{36}, G_{37} \} \\
\mathcal{T}_{11} &= \{ G_{38}, G_{39} \} \\
\mathcal{T}_{12} &= \{ G_{40} \} \\
\mathcal{T}_{13} &= \{ G_{41} \} \\
\mathcal{T}_{14} &= \{ G_{42} \} \\
\mathcal{T}_{15} &= \{ G_{43} \} \\
\mathcal{T}_{16} &= \{ G_{44} \}
\end{aligned}
$$

We also calculate the values of $\alpha'(G_i, 4)$ by using Lemma 2.3 and we list them in column four of Table 1 [12]. We now present our main result in the following theorem.

Theorem 3.1 *Every graph in $\mathcal{K}_2^{-6}(p,q)$ with $p > q = 4$ is χ-unique if one of the following conditions is satisfied:*

(i) $p \geq 7$,

(ii) $p = 6$ *and* $\Delta(G') \geq 5$.

Proof Observe that for any i, j with $1 \leq i < j \leq 16$, $\alpha'(G,3) < \alpha'(H,3)$ if $G \in \mathcal{T}_i$ and $H \in \mathcal{T}_j$. Thus by Lemma 2.1 and Equation (1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 16)$ are $\chi-$disjoint and since $\mathcal{K}_2^{-6}(p,4)$ is $\chi-$closed under the conditions (iii) or (iv) of Theorem 2.1, then each $\mathcal{T}_i$ $(1 \leq i \leq 16)$ is $\chi-$closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are $\chi-$unique, it suffices to show that for any two graphs, G, $H \in \mathcal{T}_i$, if $G \ncong H$, then either $\alpha'(G,4) \neq \alpha'(H,4)$ or $\alpha(G,5) \neq \alpha(H,5)$. Note that $\mathcal{T}_1$, $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_{14}$, $\mathcal{T}_{15}$ and $\mathcal{T}_{16}$ contains only one graph G_1, G_{41}, G_{42}, G_{43} and G_{44}, respectively and hence G_1, G_{40}, G_{41}, G_{42}, G_{43} and G_{44} are $\chi-$unique. The remaining work is to compare every two graphs in $\mathcal{T}_i$ for $2 \leq i \leq 11$. Note that all graphs in $\mathcal{T}_i$ $(2 \leq i \leq 11)$ are not considerable for the case $p = 6$ since $\Delta(G') < 5$. Thus, for all $\mathcal{T}_i$, we only consider the case $p \geq 7$.

[1] $\mathcal{T}_2$

$$
\begin{aligned}
&\alpha'(G_4,4) - \alpha'(G_2,4) \\
&= \left[3 \cdot 2^{p-3} + 3 \cdot 2^{q-2} + 21 \right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 14 \right]
\end{aligned}
$$

$$= -2^{p-3} + 2^{q-3} + 7 < 0,$$

$$\alpha'(G_2, 4) - \alpha'(G_3, 4)$$
$$= \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 14 \right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 18 \right]$$
$$= -4 < 0,$$

$$\alpha'(G_3, 4) - \alpha'(G_5, 4)$$
$$= \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 18 \right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 21 \right]$$
$$= -3 < 0.$$

Thus, we can conclude that $\alpha'(G_i, 4) \neq \alpha'(G_j, 4)$ for $2 \leq i < j \leq 5$.

[2] $\mathcal{T}_3$

$$\alpha'(G_{11}, 4) - \alpha'(G_7, 4)$$
$$= \left[7 \cdot 2^{p-4} + 4 \cdot 2^{q-2} + 16 \right] - \left[5 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 18 \right]$$
$$= -3 \cdot 2^{p-4} + 2^{q-3} - 2 < 0,$$

$$\alpha'(G_7, 4) - \alpha'(G_6, 4)$$
$$= \left[5 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 18 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 14 \right]$$
$$= -2^{p-3} + 2^{q-3} + 4 < 0,$$

$$\alpha'(G_6, 4) - \alpha'(G_8, 4)$$
$$= \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 14 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] = -4 < 0,$$

$$\alpha'(G_8, 4) - \alpha'(G_9, 4)$$
$$= \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] = 0,$$

$$\alpha'(G_8, 4) - \alpha'(G_{10}, 4)$$
$$= \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 19 \right] = -1 < 0.$$

Thus, we can conclude that $\alpha'(G_i, 4) \neq \alpha'(G_j, 4)$ for $6 \leq i < j \leq 12$ except for the graphs G_8 and G_9. Since $\alpha'(G_8, 4) = \alpha'(G_9, 4)$, we need to compare $\alpha(G_8, 5)$ and $\alpha(G_9, 5)$. By using Lemma 2.4, we can show that $\alpha(G_8, 5) \neq \alpha(G_9, 5)$ (see [12]).

[3] $\mathcal{T}_4$

$$\alpha'(G_{15},4) - \alpha'(G_{17},4)$$
$$= \left[9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 21\right] - \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 12\right]$$
$$= -2^{p-3} + 2^{q-3} + 9 < 0,$$

$$\alpha'(G_{17},4) - \alpha'(G_{16},4)$$
$$= \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 12\right] - \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 21\right] = -9 < 0,$$

$$\alpha'(G_{16},4) - \alpha'(G_{14},4)$$
$$= \left[11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 21\right] - \left[7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 11\right]$$
$$= -3 \cdot 2^{p-4} + 2 \cdot 2^{q-3} + 10 < 0,$$

$$\alpha'(G_{14},4) - \alpha'(G_{12},4)$$
$$= \left[7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 11\right] - \left[7 \cdot 2^{p-3} + 8 \cdot 2^{q-3} + 15\right]$$
$$= -2^{q-3} - 4 < 0,$$

$$\alpha'(G_{12},4) - \alpha'(G_{13},4)$$
$$= \left[7 \cdot 2^{p-3} + 8 \cdot 2^{q-3} + 15\right] - \left[7 \cdot 2^{q-3} + 8 \cdot 2^{p-3} + 15\right]$$
$$= -2^{p-3} + 2^{q-3} < 0.$$

Thus, we can conclude that $\alpha'(G_i,4) \neq \alpha'(G_j,4)$ for $13 \leq i < j \leq 18$.

[4] $\mathcal{T}_5$: We consider two cases $p = 7$ and $p \geq 8$.

(4.1) **Case 1:** When $p = 7$.

$$\alpha'(G_{25},4) - \alpha'(G_{23},4)$$
$$= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 12\right] - \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 18\right] - 6 < 0,$$

$$\alpha'(G_{23},4) - \alpha'(G_{22},4)$$
$$= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 18\right] - \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 8\right]$$
$$= -2 \cdot 2^{p-4} + 2^{q-3} + 10 < 0,$$

$$\alpha'(G_{22},4) - \alpha'(G_{21},4)$$
$$= \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 8\right] - \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 24\right]$$
$$= -2 \cdot 2^{p-4} - 2^{q-3} - 16 < 0,$$

$$\alpha'(G_{21},4) - \alpha'(G_{24},4)$$

$$= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 24 \right] - \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 18 \right]$$
$$= -2 \cdot 2^{p-4} + 2^{q-3} + 6 < 0,$$

$$\alpha'(G_{24}, 4) - \alpha'(G_{18}, 4)$$
$$= \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 18 \right] - \left[8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17 \right]$$
$$= -2^{p-4} + 2^{q-3} + 1 < 0,$$

$$\alpha'(G_{18}, 4) - \alpha'(G_{20}, 4)$$
$$= \left[8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17 \right] - \left[9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 12 \right]$$
$$= -2^{p-3} + 5 < 0,$$

$$\alpha'(G_{20}, 4) - \alpha'(G_{19}, 4)$$
$$= \left[9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 12 \right] - \left[8 \cdot 2^{q-3} + 9 \cdot 2^{p-3} + 17 \right]$$
$$= 2^{q-3} - 5 < 0.$$

Thus, we have $\alpha'(G_{25}, 4) < \alpha'(G_{23}, 4) < \alpha'(G_{22}, 4) < \alpha'(G_{21}, 4) < \alpha'(G_{24}, 4) < \alpha'(G_{18}, 4) < \alpha'(G_{20}, 4) < \alpha'(G_{19}, 4)$.

(4.2) **Case 2:** When $p \geq 8$, we can easily show that $\alpha'(G_{25}, 4) < \alpha'(G_{23}, 4) < \alpha'(G_{21}, 4) < \alpha'(G_{22}, 4) < \alpha'(G_{24}, 4) < \alpha'(G_{18}, 4) < \alpha'(G_{20}, 4) < \alpha'(G_{19}, 4)$.

Thus, we conclude that $\alpha'(G_i, 4) \neq \alpha'(G_j, 4)$ for $18 \leq i < j \leq 25$.

Similarly, we can show that for any two graphs, $G, H \in \mathcal{T}_i$ $(6 \leq i \leq 11)$, then $\alpha'(G, 4) \neq \alpha'(H, 4)$. For details, see [12]. Hence, the proof of the theorem is now completed. $\square$

In view of Theorem 3.1 and results in [9], we posed the following problem:

Problem. Study the chromaticity of any graph in $\mathcal{K}_2^{-6}(p, q)$ with $p > q$ and $q = 5, 6$.

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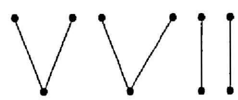
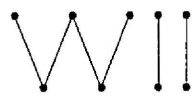
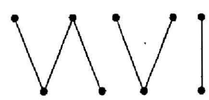
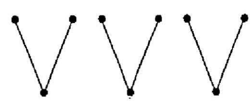
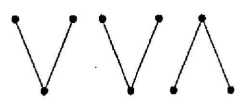
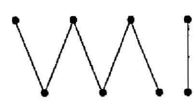
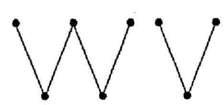
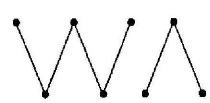
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,4} - G_i$) $\lvert A'\rvert = p$, $\lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_1	 A' B'	8	$2 \cdot 2^{p-3} + 2 \cdot 2^{q-2} + 18$
G_2	 A' B'	9	$4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 14$
G_3	 A' B'	9	$4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 18$
G_4	 A' B'	9	$3 \cdot 2^{p-3} + 3 \cdot 2^{q-2} + 21$
G_5	 A' B'	9	$4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 21$
G_6	 A' B'	10	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 14$
G_7	 A' B'	10	$5 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 18$
G_8	 A' B'	10	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18$

TABLE 1 (1 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

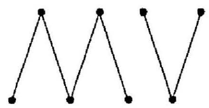
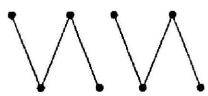
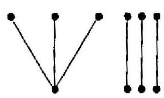
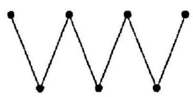
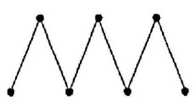
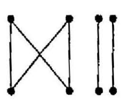
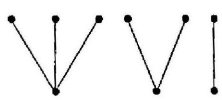
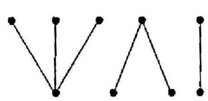
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p$, $\lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_9	 A' B'	10	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18$
G_{10}	 A' B'	10	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 19$
G_{11}	 A' B'	10	$7 \cdot 2^{p-4} + 4 \cdot 2^{q-2} + 16$
G_{12}	 A' B'	11	$7 \cdot 2^{p-3} + 8 \cdot 2^{q-3} + 15$
G_{13}	 A' B'	11	$8 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 15$
G_{14}	 A' B'	11	$7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 11$
G_{15}	 A' B'	11	$9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 21$
G_{16}	 A' B'	11	$11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 21$

TABLE 1 (2 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

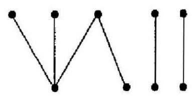
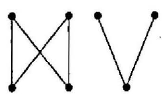
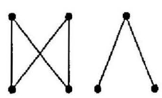
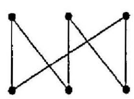
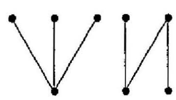
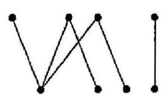
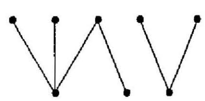
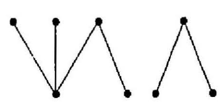
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,4} - G_i$) $\lvert A'\rvert = p$, $\lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{17}	 A' B'	11	$11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 12$
G_{18}	 A' B'	12	$8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17$
G_{19}	 A' B'	12	$8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17$
G_{20}	 A' B'	12	$9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 12$
G_{21}	 A' B'	12	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 24$
G_{22}	 A' B'	12	$15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 8$
G_{23}	 A' B'	12	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 18$
G_{24}	 A' B'	12	$15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 18$

TABLE 1 (3 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

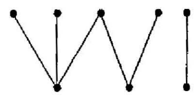
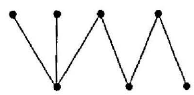
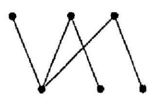
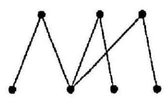
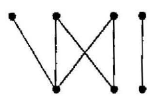
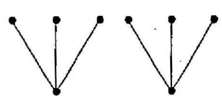
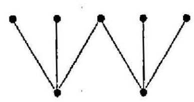
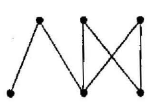
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,4} - G_i$) $\lvert A'\rvert = p$, $\lvert B'\rvert = 4$	$\alpha'(G_i,3)$	$\alpha'(G_i,4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{25}	 A' B'	12	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 12$
G_{26}	 A' B'	13	$17 \cdot 2^{p-4} + 12 \cdot 2^{q-3} + 15$
G_{27}	 A' B'	13	$17 \cdot 2^{p-4} + 12 \cdot 2^{q-3} + 9$
G_{28}	 A' B'	13	$19 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 4$
G_{29}	 A' B'	14	$9 \cdot 2^{p-3} + 11 \cdot 2^{q-3} + 6$
G_{30}	 A' B'	14	$14 \cdot 2^{p-4} + 8 \cdot 2^{q-2} + 33$
G_{31}	 A' B'	15	$18 \cdot 2^{p-4} + 17 \cdot 2^{q-3} + 15$
G_{32}	 A' B'	15	$23 \cdot 2^{p-4} + 14 \cdot 2^{q-3} + 1$

TABLE 1 (4 of 6): Graphs in $\mathcal{K}_2^{-6}(p,4)$, $p \geq 6$

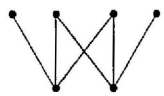
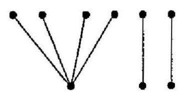
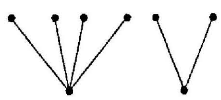
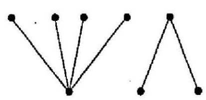
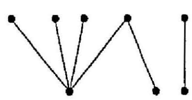
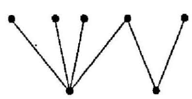
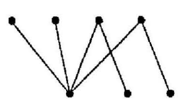
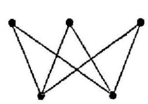
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p$, $\lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{33}	 A' B'	17	$24 \cdot 2^{p-4} + 19 \cdot 2^{q-3} - 1$
G_{34}	 A' B'	17	$33 \cdot 2^{p-5} + 11 \cdot 2^{q-2} + 9$
G_{35}	 A' B'	18	$37 \cdot 2^{p-5} + 12 \cdot 2^{q-2} + 21$
G_{36}	 A' B'	18	$12 \cdot 2^{p-2} + 37 \cdot 2^{q-5} + 21$
G_{37}	 A' B'	18	$41 \cdot 2^{p-5} + 23 \cdot 2^{q-3}$
G_{38}	 A' B'	19	$45 \cdot 2^{p-5} + 25 \cdot 2^{q-3} + 3$
G_{39}	 A' B'	19	$49 \cdot 2^{p-5} + 24 \cdot 2^{q-3} + 9$
G_{40}	 A' B'	20	$49 \cdot 2^{p-4} + 17 \cdot 2^{q-2} - 22$

TABLE 1 (5 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

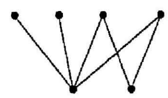
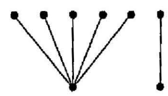
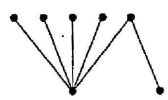
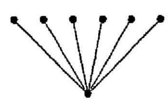
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,4} - G_i$) $\lvert A'\rvert = p$, $\lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{41}	 A' B'	21	$57 \cdot 2^{p-5} + 27 \cdot 2^{q-3} - 13$
G_{42}	 A' B'	32	$131 \cdot 2^{p-6} + 26 \cdot 2^{q-2} - 21$
G_{43}	 A' B'	33	$147 \cdot 2^{p-6} + 53 \cdot 2^{q-3} - 39$
G_{44}	 A' B'	63	$473 \cdot 2^{p-7} + 57 \cdot 2^{q-2} - 114$

TABLE 1 (6 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$